Lesson 9

We will begin this week by reading **pages 195-208** from Chapter 4 on Matrices. This lesson is divided into two sections. Each section's focus and objective will be listed for you. As you read through your text, take appropriate notes to help prepare for tests.

Section 4-7: Identity and Inverse Matrices
Objectives: Determine whether two matrices are inverse. Find the inverse of a 2x2 matrix.

Section 4-8: Using Matrices to Solve Systems of Equations
Objectives: Write matrix equations for systems of equations. Solve systems of equations using matrix equations.

After you finish reading this chapter, go over your notes from Chapters 1-4 one last time to prepare for the comprehensive test this week. Make sure you are clear on all focus points in Chapters 1-4 and that you feel comfortable with the material. Click on the link to complete the test. Good luck!

Lesson 9

Identity and Inverse Matrices

Vocabulary

- **identity matrix**
- **inverse**

What You'll Learn

- Determine whether two matrices are inverses.
- Find the inverse of a 2x2 matrix.

How are inverse matrices used in cryptography?

With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using the "key" to the message.

The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message would multiply by the opposite, or inverse, of the coding matrix.

Code								
_ 0	A 1	B 2	C 3	D 4	E 5	F 6	G 7	H 8
I 9	J 10	K 11	L 12	M 13	N 14	O 15	P 16	Q 17
R 18	S 19	T 20	U 21	V 22	W 23	X 24	Y 25	Z 26

IDENTITY AND INVERSE MATRICES Recall that in real numbers, two numbers are inverses if their product is the identity, 1. Similarly, for matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix. If *A* is any *nxn* matrix and *I* is the *nxn* identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

Key Concept

2×2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Two *nxn* matrices are **inverses** of each other if their product is the identity matrix. If matrix *A* has an inverse symbolized by A^{-1}, then A^{-1},

Example 1

Verify Inverse Matrices

Determine whether each pair of matrices are inverses.

Study Tip

a. $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ **and** $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

Check to see if $X \cdot Y = I$.

$$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 1-2 & 1+\frac{1}{2} \\ -\frac{1}{2}+(-4) & -\frac{1}{2}+1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ Matrix multiplication

Since $X \cdot Y \neq I$, they are *not* inverses.

b. $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Find *P*•*Q*.

$$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Now find *Q*•*P*

$$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 3-2 & 4-4 \\ -\frac{3}{2}+\frac{3}{2} & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Since *P*•*Q* = *Q*•*P* = *I*, *P* and *Q* are inverses.

FIND INVERSE MATRICES Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

Key Concept

Notice that *ad*-*bc* is the value of det *A*. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

Example 2

Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

Extra Examples

a. $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

Find the value of the determinant.

$$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$$

Since the determinant equals 0, R^{-1} does not exist.

b. $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the value of the determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{3(2) - 1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad a = 3,\ b = 1,\ c = 5,\ d = 2$$

$$= 1\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

CHECK

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & 2-2 \\ -15+15 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ \surd$$

Matrices can be used to code messages by placing the message in a 2x*n* matrix.

Example 3

Use Inverses to Solve a Problem

More About . . .

Cryptography

The Enigma was a German coding machine used in World War II. Its code was considered to be unbreakable. However, the code was eventually solved by a group of Polish mathematicians.
Source: www.bletchleypark.org.uk

a. **CRYPTOGRAPHY Use the table at the beginning of the lesson to assign a number to each letter**

in the message GO_TONIGHT. Then code the message with the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$**.**

Convert the message to numbers using the table.

G	O	_	T	O	N	I	G	H	T
7	15	0	20	15	14	9	7	8	20

Write the message in matrix form. Then multiply the message matrix *B* by the coding matrix *A*.

$$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 14+60 & 7+45 \\ 0+80 & 0+60 \\ 30+56 & 15+42 \\ 18+28 & 9+21 \\ 16+80 & 8+60 \end{bmatrix}$$ Matrix multiplication

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix}$$ Simplify.

The coded message is 74|52|80|60|86|57|46|30|96|68.

Study Tip

b. **Use the inverse matrix** A^{-1} **to decode the message in Example 3*a*.**

First find the inverse matrix of $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{2(3)-(1)(4)}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$ $a=2,\ b=1,\ c=4,\ d=3$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$ Simplify.

Next, decode the message by multiplying the coded matrix *C* by A^{-1}.

$$CA^{-1} = \begin{bmatrix} 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 111-104 & -37+52 \\ 120-120 & -40+60 \\ 129-114 & -43+57 \\ 69-60 & -23+30 \\ 144-136 & -48+68 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix}$$

Use the table again to convert the numbers to letters. You can now read the message.

7	15	0	20	15	14	9	7	8	20
G	O	_	T	O	N	I	G	H	T

Check for Understanding

Concept Check

1. **Write** the 4x4 identity matrix.
2. **Explain** how to find the inverse of a 2x2 matrix.
3. **OPEN ENDED** Create a square matrix that does not have an inverse.

Guided Practice

Determine whether each pair of matrices are inverses.

Self Check Quiz

4. $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$
5. $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

6. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$
7. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$

Application

9. **CRYPTOGRAPHY** Select a headline from a newspaper or the title of a magazine article and code it using your own coding matrix. Give your message and the coding matrix to a friend to decode. (*Hint*: Use a coding matrix whose determinant is 1 and that has all positive elements.)

Practice and Apply

Homework Help

For Exercises	See Examples
10–19, 32, 33	1
20–31	2
34–41	3

Extra Practice

See page 836.

Determine whether each pair of matrices are inverses.

10. $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

11. $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$

13. $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

14. $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

15. $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Determine whether each statement is *true* or *false*.

16. Only square matrices have multiplicative identities.
17. Only square matrices have multiplicative inverses.
18. Some square matrices do not have multiplicative inverses.
19. Some square matrices do not have multiplicative identities.

Find the inverse of each matrix, if it exists.

20. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

24. $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

25. $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

26. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

27. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

28. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

29. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

30. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

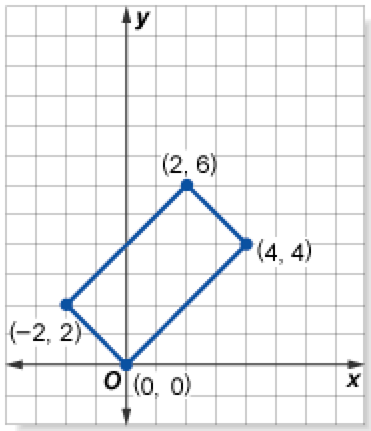
31. $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

32. Compare the matrix used to reflect a figure over the *x*-axis to the matrix used to reflect a figure over the *y*-axis.
    a. Are they inverses?
    b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

33. The matrix used to rotate a figure 270º counterclockwise about the origin is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Compare this matrix with the matrix used to rotate a figure 90º counterclockwise about the origin.
    a. Are they inverses?

b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

GEOMETRY For Exercises 34-38, use the figure below.



34. Write the vertex matrix *A* for the rectangle.
35. Use matrix multiplication to find *BA* if $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
36. Graph the vertices of the transformed rectangle. Describe the transformation.
37. Make a conjecture about what transformation B^{-1} describes on a coordinate plane.
38. Test your conjecture. Find B^{-1} and multiply it by the result of *BA*. Make a drawing to verify your conjecture.

CRYPTOGRAPHY For Exercises 39-41, use the alphabet table below.

Your friend has sent you a series of messages that were coded with the coding matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the inverse of matrix *C* to decode each message.

Code		
A 26	J 17	S 8
B 25	K 16	T 7
C 24	L 15	U 6
D 23	M 14	V 5
E 22	N 13	W 4
F 21	O 12	X 3
G 20	P 11	Y 2
H 19	Q 10	Z 1
I 18	R 9	_ 0

39. 50|36|51|29|18|18|26|13|33|26|44|22|48|33|59|34|61|35|4|2
40. 59|33|8|8|39|21|7|7|56|37|25|16|4|2
41. 59|34|49|31|40|20|16|14|21|15|25|25|36|24|32|16

42. **RESEARCH** Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded.

43. **CRITICAL THINKING** For which values of *a*, *b*, *c*, and *d* will $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$?

44. **WRITING IN MATH**

Answer the question that was posed at the beginning of the lesson.

How are inverse matrices used in cryptography?

Include the following in your answer:

- an explanation of why the inverse matrix works in decoding a message, and
- a description of the conditions you must consider when writing a message in matrix form.

Standardized Test Practice

45. What is the inverse of $\begin{bmatrix} 4 & 1 \\ 10 & 2 \end{bmatrix}$?

A. $\begin{bmatrix} -1 & \frac{1}{2} \\ 5 & -2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & -1 \\ -10 & 4 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 5 \\ \frac{1}{2} & 2 \end{bmatrix}$

D. $\begin{bmatrix} -2 & \frac{1}{2} \\ 5 & -1 \end{bmatrix}$

46. Which matrix does *not* have an inverse?

A. $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

C. $\begin{bmatrix} -3 & -3 \\ 6 & -6 \end{bmatrix}$

D. $\begin{bmatrix} -10 & -5 \\ 8 & 4 \end{bmatrix}$

Graphing Calculator Investigation

INVERSE FUNCTION The x^{-1} key on a Plus is used to find the inverse of a matrix. If you get a **SINGULAR MATRIX** error on the screen, then the matrix has no inverse.

Use a graphing calculator to find the inverse of each matrix.

47. $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$

48. $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$

49. $\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$

50. $\begin{bmatrix} 25 & -4 \\ -35 & 6 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 4 & 1 \\ 6 & 3 & 3 \end{bmatrix}$

52. $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$

Maintain Your Skills

Mixed Review

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

53. $3x+2y=-2$
$x-3y=14$

54. $2x+5y=35$
$7x-4y=-28$

55. $4x-3z=-23$
$-2x-5y+z=-9$
$y-z=3$

Evaluate each determinant by using diagonals or expansion by minors. *(Lesson 4-5)*

56. $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$

57. $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$

58. $\begin{vmatrix} 5 & -7 & 3 \\ -1 & 2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$

Find the slope of the line that passes through each pair of points. *(Lesson 2-3)*

59. (2,5), (6,9)
60. (1,0), (-2,9)

61. (-5,4), (-3,-6)
62. (-2,2), (-5,1)
63. (0,3), (-2,-2)
64. (-8,9), (0,6)

65. **OCEANOGRAPHY** The deepest point in any ocean, the bottom of the Mariana Trench in the Pacific Ocean, is 6.8 miles below sea level. Water pressure in the ocean is represented by the function $f(x)=1.15x$, where x is the depth in miles and $f(x)$ is the pressure in tons per square inch. Find the water pressure at the deepest point in the Mariana Trench. *(Lesson 2-1)*

Evaluate each expression. (Lesson 1-1)

66. $3(2^3+1)$
67. $7-5\div2+1$
68. $\frac{9-4\cdot3}{6}$
69. $[40-(7+9)]\div8$
70. $[(-2+8)6+1]8$
71. $(4-1)(8+2)^2$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each equation. (To review **solving multi-step equations**, see Lesson 1-3.)

72. $3k+8=5$
73. $12=-5h+2$
74. $7z-4=5z+8$
75. $\frac{x}{2}+5=7$
76. $\frac{3+n}{6}=-4$
77. $6=\frac{s-8}{-7}$

Using Matrices to Solve Systems of Equations

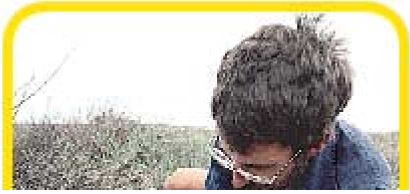
Vocabulary

- **matrix equation**

What You'll Learn

- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

How can matrices be used in population ecology?



Population ecology is the study of a species or a group of species that inhabits the same area. A biologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds during their nesting season. Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.

WRITE MATRIX EQUATIONS The situation above can be represented using a system of equations that can be solved using matrices. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$\begin{matrix} 5x+7y=11 \\ 3x+8y=18 \end{matrix} \rightarrow \begin{bmatrix} 5x+7y \\ 3x+8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

Write the matrix on the left as the product of the coefficients and the variables.

$$\underbrace{\overset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix}}}_{\text{coefficient matrix}} \cdot \underbrace{\overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}}}_{\text{variable matrix}} = \underbrace{\overset{B}{\begin{bmatrix} 11 \\ 18 \end{bmatrix}}}_{\text{constant matrix}}$$

The system of equations is now expressed as a **matrix equation**.

Example 1

Two-Variable Matrix Equation

Write a matrix equation for the system of equations.

$5x-6y=-47$
$3x+2y=-17$

Determine the coefficient, variable, and constant matrices.

$$\begin{matrix} 5x-6y=-47 \\ 3x+2y=-17 \end{matrix} \rightarrow \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

Write the matrix equation.

$$\overset{A}{\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} -47 \\ -17 \end{bmatrix}}$$

You can use a matrix equation to determine the weight of an atom of an element.

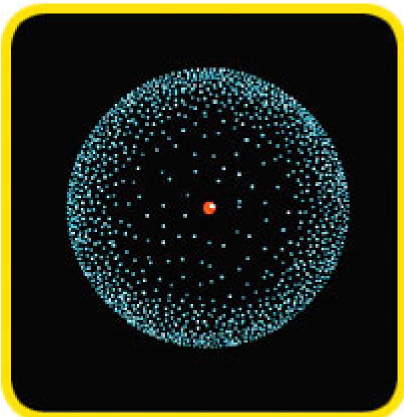
Extra Examples

Example 2

Solve a Problem Using a Matrix Equation

More About . . .

Chemistry



Atomic mass units are relative units of weight because they were compared to the weight of a hydrogen atom. So a molecule of nitrogen, whose weight is 14.0 amu, weighs 14 times as much as a hydrogen atom.
Source: www.sizes.com

CHEMISTRY The molecular formula for glucose is $C_6H_{12}O_6$, which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 atomic mass units. The formulas and weights for glucose and another sugar, sucrose, are listed below.

Sugar	Formula	Atomic Weight (amu)
glucose	$C_6H_{12}O_6$	180
sucrose	$C_{12}H_{22}O_{11}$	342

a. **Write a system of equations that represents the weight of each atom.**
Let *c* represent the weight of a carbon atom.
Let *h* represent the weight of a hydrogen atom.
Write an equation for the weight of each sugar. The subscript represents how many atoms of each element are in the molecule.

Glucose: $6c + 12h + 6(16) = 180$ Equation for glucose
$6c + 12h + 96 = 180$ Simplify.
$6c + 12h = 84$ Subtract 96 from each side.

Sucrose: $12c + 22h + 11(16) = 342$ Equation for sucrose
$12c + 22h + 176 = 342$ Simplify.
$12c + 22h = 166$ Subtract 176 from each side.

b. **Write a matrix equation for the system of equations.**

Determine the coefficient, variable, and constant matrices.

$$\begin{matrix} 6c+12h=84 \\ 12c+22h=166 \end{matrix} \rightarrow \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \begin{bmatrix} c \\ h \end{bmatrix} \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

Write the matrix equation.

$$\begin{matrix} A & \cdot & X & = & B \end{matrix}$$

$$\begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$ You will solve this matrix equation in Exercise 11.

SOLVE SYSTEMS OF EQUATIONS You can solve a system of linear equations by solving a matrix equation. A matrix equation in the form $AX=B$, where A is a coefficient matrix, X is a variable matrix, and B is a constant matrix, can be solved in a similar manner as a linear equation of the form $ax=b$.

$ax=b$ Write the equation.

$\left(\frac{1}{a}\right)ax=\left(\frac{1}{a}\right)b$ Multiply each side by the inverse of the coefficient, if it exists.

$1x=\left(\frac{1}{a}\right)b$ $\left(\frac{1}{a}\right)a=1,\ A^{-1}A=I$

$x=\left(\frac{1}{a}\right)b$ $1x=x,\ IX=X$

$AX=B$ Write the equation.

$A^{-1}AX=A^{-1}B$ Multiply each side by the inverse of the coefficient, if it exists.

$IX=A^{-1}B$ $\left(\frac{1}{a}\right)a=1,\ A^{-1}A=I$

$X=A^{-1}B$ $1x=x,\ IX=X$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

Example 3

Solve a System of Equations

Use a matrix equation to solve the system of equations.

$6x+2y=11$
$3x-8y=1$

The matrix equation is $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$, when $A=\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, and $B=\begin{bmatrix} 11 \\ 1 \end{bmatrix}$.

Step 1
Find the inverse of the coefficient matrix.

$A^{-1}=\frac{1}{-48-6}\begin{bmatrix} -8 & -2 \\ 3 & 6 \end{bmatrix}$ or $-\frac{1}{54}\begin{bmatrix} -8 & -2 \\ 3 & 6 \end{bmatrix}$

Step 2
Multiply each side of the matrix equation by the inverse matrix.

$$-\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}\cdot\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}\cdot\begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

Multiply each side by A^{-1}

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -90 \\ -27 \end{bmatrix}$$

Multiply matrices.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The solution is $\left(\frac{5}{3}, \frac{1}{2}\right)$. Check this solution in the original equation.

Study Tip

Example 4

System of Equations with No Solution

Use a matrix equation to solve the system of equations. $\begin{aligned} 6a - 9b &= -18 \\ 8a - 12b &= 24 \end{aligned}$

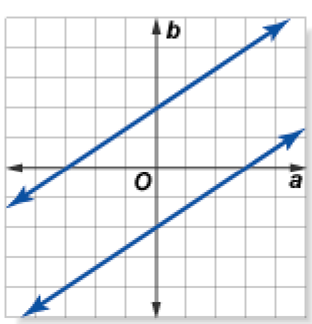
The matrix equation is $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}\cdot\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \end{bmatrix}$, and $B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$.

Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-72+72}\begin{bmatrix} -12 & 9 \\ -8 & 6 \end{bmatrix}$$

The determinant of the coefficient matrix $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ is 0, so A^{-1} does not exist.

There is no unique solution of this system.



Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.

Study Tip

To solve a system of equations with three variables, you can use the 3 X 3 identity matrix. However, finding the inverse of a 3 X 3 matrix may be tedious. Graphing calculators and computer programs offer fast and accurate methods for performing the necessary calculations.

Graphing Calculator Investigation

Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

$3x-2y+z=0$
$2x+3y-z=17$
$5x-y+4z=-7$

Think and Discuss

1. Write a matrix equation for the system of equations.
2. Enter the coefficient matrix as matrix *A* and the constant matrix as matrix *B* in the graphing calculator. Find the product of A^{-1} and *B*. Recall that the x^{-1} key is used to find A^{-1}.
3. How is the result related to the solution?

Check for Understanding

Concept Check

1. **Write** the matrix equation $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ as a system of linear equations.
2. **OPEN ENDED** Write a system of equations that does not have a unique solution.
3. **FIND THE ERROR** Tommy and Laura are solving a system of equations. They find that $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

   Tommy

   $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix}$

   $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

   Laura

   $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

   $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$

   Who is correct? Explain your reasoning.

Guided Practice

Write a matrix equation for each system of equations.

4. $\begin{aligned} x - y &= -3 \\ x + 3y &= 5 \end{aligned}$

5. $\begin{aligned} 2g + 3h &= 8 \\ -4g - 7h &= -5 \end{aligned}$

6. $\begin{aligned} 3a - 5b + 2c &= 9 \\ 4a + 7b + c &= 3 \\ 2a - c &= 12 \end{aligned}$

Solve each matrix equation or system of equations by using inverse matrices.

7. $\begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 24 \end{bmatrix}$

8. $\begin{bmatrix} 8 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \end{bmatrix}$

9. $\begin{aligned} 5x - 3y &= -30 \\ 8x + 5y &= 1 \end{aligned}$

10. $\begin{aligned} 5s + 4t &= 12 \\ 4s - 3t &= -1.25 \end{aligned}$

Application

11. **CHEMISTRY** Refer to Example 2 on page 203. Solve the system of equations to find the weight of a carbon, hydrogen, and oxygen atom.

Practice and Apply

Homework Help

For Exercises	See Examples
12–19	1
20–31	3, 4
32–34	2

Extra Practice

See page 836.

Write a matrix equation for each system of equations.

12. $\begin{aligned} 3x - y &= 0 \\ x + 2y &= -21 \end{aligned}$

13. $\begin{aligned} 4x - 7y &= 2 \\ 3x + 5y &= 9 \end{aligned}$

14. $\begin{aligned} 5a - 6b &= -47 \\ 3a + 2b &= -17 \end{aligned}$

15. $\begin{aligned} 3m - 7n &= -43 \\ 6m + 5n &= -10 \end{aligned}$

16. $\begin{aligned} 2a + 3b - 5c &= 1 \\ 7a + 3c &= 7 \\ 3a - 6b + c &= -5 \end{aligned}$

17. $\begin{aligned} 3x - 5y + 2z &= 9 \\ x - 7y + 3z &= 11 \\ 4x - 3z &= -1 \end{aligned}$

18. $\begin{aligned} x - y &= 8 \\ -2x - 5y - 6z &= -27 \\ 9x + 10y - z &= 54 \end{aligned}$

19. $\begin{aligned} 3r - 5s + 6t &= 21 \\ 11r - 12s + 16t &= 15 \\ -5r + 8s - 3t &= -7 \end{aligned}$

Solve each matrix equation or system of equations by using inverse matrices.

20. $\begin{bmatrix} 7 & -3 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$

22. $\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -17 \\ -4 \end{bmatrix}$

23. $\begin{bmatrix} 7 & 1 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 43 \\ 10 \end{bmatrix}$

24. $\begin{bmatrix} 2 & -9 \\ 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 28 \\ -12 \end{bmatrix}$

25. $\begin{bmatrix} 6 & 5 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix}$

26. $\begin{aligned} 6r + s &= 9 \\ 3r &= -2s \end{aligned}$

27. $\begin{aligned} 5a + 9b &= -28 \\ 2a - b &= -2 \end{aligned}$

28. $\begin{aligned} p - 2q &= 1 \\ p + 5q &= 22 \end{aligned}$

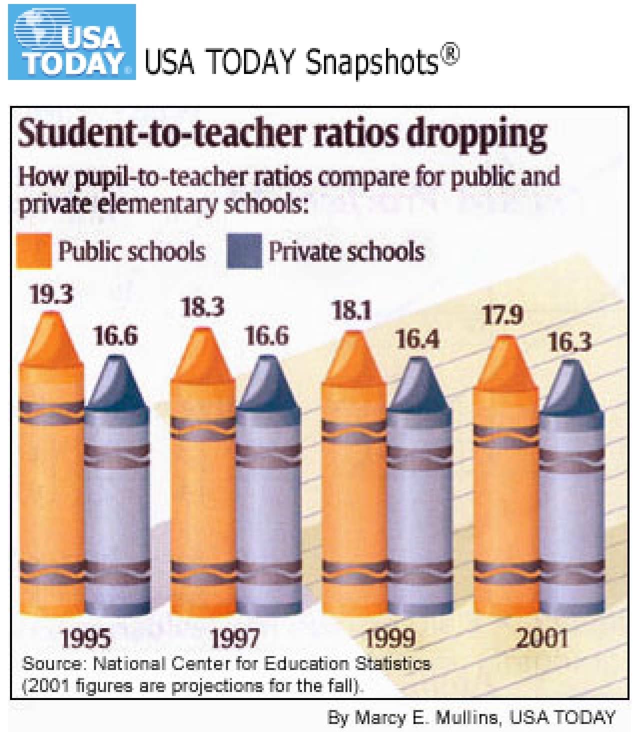
29. $\begin{aligned} 4m - 7n &= -63 \\ 3m + 2n &= 18 \end{aligned}$

30. $\begin{aligned} x + 2y &= 8 \\ 3x + 2y &= 6 \end{aligned}$

31. $\begin{aligned} 4x - 3y &= 5 \\ 2x + 9y &= 6 \end{aligned}$

32. **PILOT TRAINING** Hai-Ling is training for his pilot's license. Flight instruction costs $105 per hour, and the simulator costs $45 per hour. The school requires students to spend 4 more hours in airplane training than in the simulator. If Hai-Ling can afford to spend $3870 on training, how many hours can he spend training in an airplane and in a simulator?

33. **SCHOOLS** The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same.



34. **CHEMISTRY** Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?

Self Check Quiz

35. **CRITICAL THINKING** Describe the solution set of a system of equations if the coefficient matrix does not have an inverse.

36. WRITING IN MATH

Answer the question that was posed at the beginning of the lesson.

How can matrices be used in population ecology?

Include the following in your answer:

- a system of equations that can be used to find the number of each species the region can support, and
- a solution of the problem using matrices.

Standardized Test Practice

37. Solve the system of equations $6a+8b=5$ and $10a-12b=2$.

A. $\left(\frac{3}{4}, \frac{1}{2}\right)$

B. $\left(\frac{1}{2}, \frac{3}{4}\right)$

C. $\left(\frac{1}{2}, -\frac{1}{2}\right)$

D. $\left(\frac{1}{2}, \frac{1}{4}\right)$

38. **SHORT RESPONSE** The Yogurt Shoppe sells cones in three sizes: small $0.89; medium, $1.19; and

large, $1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold $58.98 in cones, how many of each size did he sell?

Graphing Calculator Investigation

INVERSE MATRICES Use a graphing calculator to solve each system of equations using inverse matrices.

39. $2a-b+4c=6$
$a+5b-2c=-6$
$3a-2b+6c=8$

40. $3x-5y+2z=22$
$2x+3y-z=-9$
$4x+3y+3z=1$

41. $2q+r+s=2$
$-q-r+2s=7$
$-3q+2r+3s=7$

Maintain Your Skills

Mixed Review

Find the inverse of each matrix, if it exists. *(Lesson 4-7)*

42. $\begin{bmatrix} 4 & 4 \\ 2 & 3 \end{bmatrix}$

43. $\begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$

44. $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$

Use Cramer's Rule to solve each system of equations. *(Lesson 4-6)*

45. $6x+7y=10$
$3x-4y=20$

46. $6a+7b=-10.15$
$9.2a-6b=69.944$

47. $\frac{x}{2}-\frac{2y}{3}=2\frac{1}{3}$
$3x+4y=-50$

$3x+4y=-50$

48. **ECOLOGY** If you recycle a 3½-foot stack of newspapers, one less 20-foot loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall. *(Lesson 2-5)*

Solve each equation. Check your solutions. *(Lesson 1-4)*

49. $|x-3|=7$
50. $-4|d+2|=-12$
51. $5|k-4|=k+8$

WebQuest Internet Project

Lessons in Home Buying, Selling

It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations. You may also wish to include additional data, information, or pictures.
www.algebra2.com/webquest

Graphing Calculator Investigation

A Follow-Up of Lesson 4-8

Augmented Matrices

Using a TI-83 Plus, you can solve a system of linear equations using the **MATRX** function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. The reduced row echelon function of a graphing calculator reduces the augmented matrix so that the solution of the system of equations can be easily determined.

Calculator Keystrokes

Write an augmented matrix for the following system of equations. Then solve the system by using the reduced row echelon form on the graphing calculator.

$3x+y+3z=2$
$2x+y+2z=1$
$4x+2y+5z=5$

Step 1
Write the augmented matrix and enter it into a calculator.

The augmented matrix $B=\left[\begin{array}{ccc|c} 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 5 & 5 \end{array}\right]$.

Begin by entering the matrix.

KEYSTROKES: *Review matrices on page 163.*

Step 2
Find the reduced row echelon form (rref) using the graphing calculator.

KEYSTROKES: 2nd **[MATRX]** ▶ ALPHA **[B]** 2nd **[MATRX]2**) CLEAR

Study the reduced echelon matrix. The first three columns are the same as a 3x3 identity matrix. The first row represents $x=-2$, the second row represents $y=-1$, and the third row represents $z=3$. The solution is (-2, -1, 3).

Exercises

Write an augmented matrix for each system of equations. Then solve with a graphing calculator.

1. $x-3y=5$
   $2x+y=1$

2. $15x+11y=36$
   $4x-3y=-26$

3. $2x+y=5$
   $2x-3y=1$

4. $3x-y=0$
   $2x-3y=1$

5. $3x-2y+z=-2$
   $x-y+3z=5$
   $-x+y+z=-1$

6. $x-y+z=2$
   $x-z=1$
   $y+2z=0$

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