Advanced Mathematics

Prepares the foundation for further study of mathematics at the college level through a presentation of standard pre-calculus topics, including substantial new material on discrete mathematics and data analysis. The following book is required for this course:

Saxon Advanced Mathematics: an Incremental Development (textbook and Home Study Packet), Second Edition, by John H. Saxon, Jr., Saxon Publishers Inc., 1996

Contents of *Saxon Advanced Mathematics*:

Numbers
65. Radicals in Trigonometric Equations; Graphs of Logarithmic Functions
66. Formulas for Systems of Equations; Phase Shifts and Period Changes
67. Antilogarithms
68. Locus Definition of a Parabola; Translated Parabolas; Applications; Derivation
69. Matrices; Determinants
70. Percentiles and z Scores
71. The Ellipse (1)
72. One Side Plus Two Other Parts; Law of Sines
73. Regular Polygons
74. Cramer's Rule
75. Combinations
76. Functions of ($-x$); Functions of the Other Angle; Trigonometric Identities (1); Rules of the Game
77. Binomial Expansions (1)
78. The Hyperbola
79. DeMoivre's Theorem; Roots of Complex Numbers
80. Trigonometric Identities (2)
81. Law of Cosines
82. Taking the Logarithm of; Exponential Equations
83. Simple Probability; Independent Events; Replacement
84. Factorable Expressions; Sketching Sinusoids
85. Advanced Trigonometric Equations; Clock Problems
86. Arithmetic Progressions and Arithmetic Means
87. Sum and Difference Identities; Tangent Identities
88. Exponential Functions (Growth and Decay)
89. The Ellipse (2)
90. Double-Angle Identities; Half-Angle Identities
91. Geometric Progressions
92. Probability of Either; Notations for Permutations and Combinations
93. Advanced Trigonometric Identities; Triangle Inequalities (2)
94. Graphs of Secant and Cosecant
95. Advanced Complex Roots
96. More Double-Angle Identities; Triangle Area Formula; Proof of the Law of Sines; Equal Angles Imply Proportional Sides
97. The Ambiguous Case
98. Change of Base; Contrived Logarithm Problems
99. Sequence Notations; Advanced Sequence Problems; The Arithmetic and Geometric Means
100. Product Identities; More Sum and Difference Identities
101. Zero Determinants; 3x3 Determinants; Determinant Solutions of 3x3 Systems; Independent Equations
102. Binomial Expansions (2)
103. Calculations with Logarithms; Power of the Hydrogen
104. Arithmetic Series; Geometric Series

105. Cofactors; Expansion by Cofactors
106. Translations of Conic Sections; Equations of the Ellipse; Equations of the Hyperbola
107. Convergent Geometric Series
108. Matrix Addition and Multiplication
109. Rational Numbers
110. Graphs of arcsine and arccosine; Graphs of arcsecant and arccosecant; Graphs of arctangent and arccotangent
111. Logarithmic Inequalities: Base Greater Than 1; Logarithmic Inequalities: Base Less Than 1
112. Binomial Theorem
113. Synthetic Division; Zeros and Roots
114. Graphs of Factored Polynomial Functions
115. The Remainder Theorem
116. The Region of Interest
117. Prime and Relatively Prime Numbers; Rational Roots Theorem
118. Roots of Polynomial Equations
119. Descartes's Rule of Signs Upper and Lower Bound Theorem; Irrational Roots
120. Matrix Algebra; Finding Inverse Matrices
121. Piecewise Functions; Greatest Integer Function
122. Graphs of Rational Functions; Graphs that Contain Holes
123. The General Conic Equation
124. Point of Division Formulas
125. Using the Graphing Calculator to Graph; Solutions of Systems of Equations Using the Graphing Calculator; Roots

Lesson 18

Advanced Word Problems

Word problems are worked by transforming written statements into mathematical equations and then solving the equations. Percent and ratio problems can be solved by using any one variable. Nickel and dime problems can be solved by using two variables. Many problems require the use of three or more variables for their solution. **As many variables may be used as are convenient. For a unique solution of a system of linear equations, we must have as many independent equations as we have variables.**† The variables x, y, and z should be avoided because it is difficult to remember what they represent in a particular problem. Subscripted variables should be used because their meanings are easier to remember.

Word problems tend to be categorized into types according to the different thought processes required to find their solutions. Thus far, we have looked at simple problems whose solutions required the use of at most two variables. In this lesson, we will review the solution of problems that require the use of three variables in three equations. We also will review other types of problems. Some of these problems were selected because they require procedures that have a wide variety of applications. Other problems were selected because they represent types of problems that will be encountered in almost the same forms in chemistry and physics courses.

example 18.1 The number of blues was 4 less than the sum of the whites and the greens. Also, the number of greens equaled the sum of the blues and the whites. How many of each were there if there were twice as many blues as whites?

solution This problem can be worked by using three equations in three unknowns. We will use N_B, N_W, and N_G as the variables. The three equations are as follows:

(a) The number of blues was 4 less than the sum of the whites and the greens.

$$N_B + 4 = N_W + N_G$$

(b) The number of greens equaled the sum of the blues and the whites.

$$N_G = N_B + N_W$$

(c) There were twice as many blues as whites.

$$N_B = 2N_W$$

† This is true for systems of linear equations if the domain for all variables and all coefficients is the set of real numbers.

Note that in (a) we added 4 to the number of blues because there were 4 fewer blues. Also, in (c) we multiplied the number of whites by 2 to equal the number of blues. When a statement tells how much greater or less a quantity is, addition or multiplication is required so that an equation (statement of equality) may be written. We begin by substituting $2N_W$ for N_B in equations (a) and (b).

$$\begin{array}{lcr} \text{(a) } (2N_W) + 4 = N_W + N_G & \rightarrow & N_W - N_G = -4 \\ \text{(b) } N_G = (2N_W) + N_W & \rightarrow & \underline{-3N_W + N_G = 0} \\ & & -2N_W = -4 \\ & & N_W = 2 \end{array}$$

Now N_B equals $2N_W$, so $N_B = 4$; and N_G equals $N_B + N_W$, so $N_G = 6$. Thus

$N_W = \mathbf{2}$ $\quad$ $N_B = \mathbf{4}$ $\quad$ $N_G = \mathbf{6}$

example 18.2 The quarters, nickels, and dimes totaled 20, and their value was \$1.90. How many of each kind were there if there were 4 times as many nickels as quarters?

solution There were 20 coins in all.

$$\text{(a) } N_N + N_D + N_Q = 20$$

and their value was \$1.90.

$$\text{(b) } 5N_N + 10N_D + 25N_Q = 190$$

There were 4 times as many nickels as quarters.

$$\text{(c) } N_N = 4N_Q$$

We begin by using (c) to substitute for N_N in (a) and (b).

$$\begin{array}{llcll} \text{(a)} & (4N_Q) + N_D + N_Q = 20 & \rightarrow & N_D + 5N_Q = 20 & \text{(a')} \\ \text{(b)} & 5(4N_Q) + 10N_D + 25N_Q = 190 & \rightarrow & 10N_D + 45N_Q = 190 & \text{(b')} \end{array}$$

Now we multiply (a') by -10 and add to (b').

$$\begin{array}{rr} \text{(-10)(a')} & -10N_D - 50N_Q = -200 \\ \text{(b')} & \underline{10N_D + 45N_Q = 190} \\ & -5N_Q = -10 \\ & N_Q = 2 \end{array}$$

$N_N = 4(2) = \mathbf{8}$, and since there were 20 in all, **10 were dimes**.

example 18.3 Reds varied directly as blues squared and inversely as greens. When there were 80 reds, there were 4 blues and 2 greens. How many reds were there when there were 8 blues and 10 greens?

solution The problem can be worked as a variation problem. This approach is often used in physics books. The first sentence gives us the basic equation.

$$\text{(1) R} = \frac{kB^2}{G} \qquad\qquad \text{(2) R} = \frac{kB^2}{G}$$

Writing the equation twice is a mnemonic to help us remember that this is a two-step problem. The first step is to use the numbers given in the second sentence in the left-hand equation and find that *k* equals 10. Then we replace *k* with 10 in the right-hand equation.

$$80 = \frac{k(4)^2}{2} \rightarrow k = 10 \rightarrow R = \frac{10B^2}{G}$$

To finish, we use the second set of numbers in the right-hand equation and find that *R* equals 64.

$$R = \frac{10(8)^2}{10} \rightarrow R = \mathbf{64}$$

The ratio method can also be used to work on this problem. This approach is often used in chemistry books. The first sentence gives us the basic equation.

$$\frac{R_1}{R_2} = \frac{B_1^{\,2}G_2}{B_2^{\,2}G_1}$$

Now we make the required replacements and solve.

$$\frac{80}{R_2} = \frac{(4)^2(10)}{(8)^2(2)} \rightarrow \frac{80}{R_2} = \frac{160}{128} \rightarrow R_2 = \mathbf{64}$$

example 18.4 The sum of the digits of a two-digit counting number is 5. When the digits are reversed, the number is 9 greater than the original number. What was the original number?

solution The counting numbers are the positive integers. The sum of the digits is 5. If we use *U* for the units' digit and *T* for the tens' digit, we get

$$\text{(a) } U + T = 5$$

The value of the units' digit is U and of the tens' digit is $10T$, but when the digits are reversed the values will be $10U$ and T.

	ORIGINAL NUMBER		NEW NUMBER MINUS 9
(b)	$10T + U$	=	$T + 10U - 9$

which simplifies to

$$9T - 9U = -9 \rightarrow T - U = -1$$

Now we substitute from equation (a) and solve.

$(5 - U) - U = -1$	substituted for T
$5 - 2U = -1$	added
$-2U = -6$	added -5 to both sides
$U = 3$	solved

Since $U + T = 5$, $T = 2$, and the original number was **23**.

example 18.5 To get 1000 gallons (gal) of a mixture that was 56% alcohol, it was necessary to mix a quantity of 20% alcohol solution with a quantity of 80% alcohol solution. How much of each was required?

solution We decide to make the statement about alcohol.

$$\text{Alcohol}_1 + \text{alcohol}_2 = \text{alcohol total}$$

Next, we use parentheses as mixture containers.

$$(\quad) + (\quad) = (\quad)$$

We pour in P_N gallons of the first mixture and dump in D_N gallons of the second mixture for a total of 1000 gallons.

$$(P_N) + (D_N) = (1000)$$

Now we multiply by the proper decimals so that each entry represents gallons of alcohol.

$$\text{(a) } 0.2(P_N) + 0.8(D_N) = 0.56(1000)$$

This equation has two unknowns, so we need another equation, which is

$$\text{(b) } P_N + D_N = 1000$$

Now we substitute to solve.

$0.2(1000 - D_N) + 0.8D_N = 0.56(1000)$ substituted

$200 - 0.2D_N + 0.8D_N = 560$ multiplied

Now we eliminate the decimals by multiplying by 10.

$2000 - 2D_N + 8D_N = 5600$ multiplied by 10

$6D_N = 3600$ added

$D_N =$ **600 gal of 80% alcohol** solved

Since the total was 1000 gallons, we need **400 gallons of 20% alcohol**.

example 18.6 How many liters of a 64% glycol solution must be added to 77 liters of a 23% glycol solution to get a 42% glycol solution?

solution The solution to this problem is not difficult if a calculator is used to help with the arithmetic. We will make the statement about glycol and then insert the indicated quantities in the parentheses used as mixture containers.

Glycol_1	+	glycol added	=	glycol final
(77)	+	(P_N)	=	$(77 + P_N)$

The mixture entries indicate the amount of mixture. It is important to use symbols such as P_N or D_N for the amount of solution added. **Avoid using *G* for glycol added because the mixture added was not all glycol.** Next, we multiply each of the mixture container entries by the proper decimal number so that the product will equal the amount of glycol for each step.

$$0.23(77) + 0.64(P_N) = 0.42(77 + P_N)$$

We use a calculator to permit a quick solution to this equation.

$17.71 + 0.64P_N = 32.34 + 0.42P_N$ multiplied

$0.22P_N = 14.63$ rearranged

$P_N =$ **66.5 liters** divided

example 18.7 The weight of the carbon (C) in the container of C_3H_7Cl was 113 grams. What was the total weight of the compound? (C = 12; H = 1; Cl = 35)

solution This is a ratio problem. The gram atomic weights are given above in parenthesis.

Carbon	12 x 3 =	36
Hydrogen	1 x 7 =	7
Chlorine	35 x 1 =	35
Total		78

Thus, the ratio of the carbon to the total weight is 36 to 78, and the carbon weighed 113 grams.

$$\frac{C}{T} = \frac{36}{78} \rightarrow \frac{113}{T} \rightarrow \frac{36}{78} \quad T = \mathbf{244.83\ grams}$$