Math Connections

Sample

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INDEPENDENT LEARNING SINCE 1975

Math Connections

Oak Meadow
Coursebook

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Lesson 3 Number Theory and the Real Number System

Now that we have learned how to classify objects into sets, we can explore how to classify numbers. The number 3, for example, may seem like "just a number." However, it fits into many categories of numbers, including the natural numbers, the whole numbers, the integers, the rational numbers, the real numbers, the prime numbers, and the triangular numbers. In this lesson, we will examine the primary classifications of numbers and we will also explore some unique numerical properties.

This lesson should take approximately three weeks to complete.

Learning Objectives

- Determine whether a natural number is prime or composite.
- Apply divisibility rules to natural numbers.
- Find the greatest common factor and prime factorization for a number.
- Find the least common multiple for two numbers.
- Apply the order of operations.
- Perform operations and solve problems with rational numbers.
- Simplify and perform operations with square roots.
- Classify numbers into sets and subsets.
- Convert between decimal and scientific notation.
- Write terms of arithmetic and geometric sequences.
- Explore recent mathematical discoveries of prime numbers.
- Identify prime numbers and recognize patterns using the Sieve of Eratosthenes.
- Explore some real-world applications of the Fibonacci sequence.

ASSIGNMENT SUMMARY

- ☐ Mental Math Set A: Divisibility Rules
- ☐ Mental Math Set B: Exponents
- ☐ Mental Math Set C: Scientific Notation
- ☐ Read Chapter 3 in textbook.
- ☐ Complete a selection of exercises for sections 3.1 through 3.7.
- ☐ Read Chapter 3 Summary.
- ☐ Complete test from textbook OR test packet.
- ☐ Math Journal A: Divisibility Trick for Seven
- ☐ Math Journal B: The Largest Known Prime Number (so far)
- ☐ Math Journal C: Fibonacci Connections
- ☐ Activity A: The Sieve of Eratosthenes
- ☐ Activity B: Fibonacci Inspiration

Number Theory and the Real Number System

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"Numbers are the highest degree of knowledge. It is knowledge itself."

Plato

Why It Matters

Studying patterns, classification, and structure of numbers allows us to make sense of them and to apply them in useful ways. For example, breaking numbers down into their prime factorization lets us easily find their least common multiple. This may seem like an academic triviality, but it can have very useful applications. Consider, for example, that two events each happen on a particular schedule, say every 30 days and every 42 days, and you need to determine the next time that they will occur on the same day. Breaking down the numbers using prime factorization and then finding their least common multiple will help you easily solve this problem.

Here is another example: Studying prime numbers may seem like something useful only in math class, but prime numbers are actually at work behind the scenes every time you enter a computer password or make an online purchase. Prime numbers are used by computer programmers to encrypt your data so it remains secure. There really is a practical use for the frenzy behind the quest to find ever-larger prime numbers!

Mental Math Warms-ups

This lesson contains three sets of mental math warm-ups. Complete one set each week.

- Mental Math Set A: Divisibility Rules
- Mental Math Set B: Exponents
- Mental Math Set C: Scientific Notation

Mental Math Set A: Divisibility Rules

Is the number 1,290,354,042 divisible by three? If your first inclination is to grab a pen and paper to start doing long division, wait a minute! There is an easier way to tell if a number is divisible by small numbers.

You are probably already familiar with some divisibility rules. If a number is even, then it is divisible by two. If a number ends in a zero, then it is divisible by 10. And if a number ends in either 5 or 0, then that number is divisible by 5. But have you considered *why* these facts are true?

In order for a number to be divisible by a second number, the original number must contain the second number as a *factor*. (A factor is simply a number that is multiplied by other numbers.) For example, the number 90, which we know is divisible by 10, can be written as 9x10. Since 10 is a factor of 90, 90 is divisible by 10. Likewise, 9 is a factor of 90, so 90 is also divisible by 9. And since we could also write 90 as the product 2×3×3×5, we can see that 2, 3, and 5 are also factors!

In this week's mental math exercises, we will explore some useful divisibility rules that aren't as commonly known. In your textbook, you'll find a handy reference guide for the divisibility rules (Table 3.1).

Day 1: Practice testing numbers for divisibility by 3. With a partner, take turns giving each other a number and determining whether it is divisible by 3. (Remember, a number is divisible by 3 if the sum of its digits is divisible by 3.) Gradually increase the difficulty by calling out numbers with more digits.

Day 2: Practice testing numbers for divisibility by 4. With a partner, take turns giving each other a number and determining whether it is divisible by 4. (Remember, a number is divisible by 4 if its last two digits are divisible by 4.) Gradually increase the difficulty.

Day 3: Practice testing numbers for divisibility by 6. With a partner, take turns giving each other a number and determining whether it is divisible by 6. (Remember, a number is divisible by 6 if it is divisible by both 2 and 3.) Gradually increase the difficulty.

Day 4: Practice testing numbers for divisibility by 9. With a partner, take turns giving each other a number and determining whether it is divisible by 9. (Remember, a number is divisible by 9 if the sum of its digits is divisible by 9.) Gradually increase the difficulty.

Day 5: Practice testing numbers for divisibility by 12. With a partner, take turns giving each other a number and determining whether it is divisible by 12. (Remember, a number is divisible by 12 if is divisible by both 3 and 4.) Gradually increase the difficulty.

Mental Math Set B: Exponents

Exponents are a regular part of algebra and geometry, so having common squares and cubes memorized will help you work more quickly and accurately.

Number Theory and the Real Number System

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Day 1: Memorize the perfect squares up to 15^2. With a partner, quiz each other, gradually increasing speed, until you can both quickly recall all of them.

Day 2: Memorize the perfect cubes up to 10^3. With a partner, quiz each other, gradually increasing speed, until you can both quickly recall all of them.

Day 3: Practice estimating the value of square roots by determining two numbers that the square root must fall between. For example, we can estimate that $\sqrt{48}$ is somewhere between 6 and 7 because 48 is between the perfect squares 36 and 49, which have square roots 6 and 7, respectively. With a partner, alternate giving each other square root exercises, and estimating the values.

Days 4 and on: Repeat the exercise for Day 3, but this time try to get even closer to the actual value by estimating how close the number will be to each of the boundary values. For example, 48 is much closer to 49 than to 36, so we can infer that $\sqrt{48}$ will be closer to 7 than to 6. That makes 6.9 a better estimate than, say, 6.2. Challenge your partner to see who can get closer to the actual value of each of your exercises. Use a calculator to check the accuracy of your estimates.

Mental Math Set C: Scientific Notation

Scientific notation is used so often that it is very helpful to be fluent in converting numbers from decimal notation to scientific notation and back again.

Days 1 and 2: Practice mentally converting back and forth from decimal notation to scientific notation by shifting the decimal point accordingly. For example, $3.6\times10^4=36000$ because the exponent of +4 indicates that we move the decimal point 4 places to the right. Likewise, $6.5\times10^{-5}=0.000065$ because the –5 in the exponents tells us to move the decimal point 5 places to the left. Make up several examples, either alone or with a friend, and mentally convert from one system of notation to the other.

Days 3 and on: Practice mentally simplifying products and quotients involving scientific notation. Use estimation techniques and the commutative property of multiplication to rearrange the numbers in order to easily simplify problems. For example, a good estimate for

$(5.1\times10^{3})(6.9\times10^{-2})$ is $(5\times10^{3})(7\times10^{-2})$. We can then change the order of the factors, like this:

$$(5\times7)(10^{3}\times10^{-2})=35\times10^{1}$$

That gives us an estimated value of about 350. Make up several of your own problems to mentally estimate and solve.

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"Perfect numbers like perfect men are very rare."

René Descartes

Lesson 3 Assignments

Textbook Assignments and Test

1. Read textbook sections 3.1 through 3.7. For each section, follow along with the examples and try the Checkpoint problems. Check your answers with the back of the book. Verbally answer the Concept and Vocabulary Check exercises at the end of the section and check your answers with the back of the book.
2. After reading each textbook section, complete a selection of problems from each section of the Practice Exercises 3.1 through 3.7 (odd-numbered problems only). Choose several problems of each type to ensure sufficient practice.
3. Do the following Application Exercises:
    - Exercise Set 3.1: Application Exercises 91 and 95.
    - Exercise Set 3.2: Application Exercises 115, 127, and 129.
    - Exercise Set 3.3: Application Exercises 117, 119, 121, 123, 127, 131, and 133.
    - Exercise Set 3.4: Application Exercises 75, 77, and 79.
    - Exercise Sets 3.5 through 3.7: do all Application Exercises (odds only).

    Check your answers with the back of the book. Make any necessary corrections and review areas that need work. Feel free to complete a selection of even-numbered problems for extra practice.

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4. Review the Chapter 3 Summary at the end of the chapter. Select problems from the Chapter 3 Review at the end of the chapter if you feel you need additional practice.
5. Complete the Chapter 3 Test from the textbook (for independent students) or the Lesson 3 Test from the test packet (for enrolled students). Students who complete the textbook test can check their answers in the back of the book, making necessary corrections and reviewing areas that need work. **Students who are enrolled in Oak Meadow School must complete the test from the test packet.**

Math Journal

Complete all three journal assignments (do one per week).

- Math Journal A: Divisibility Trick for Seven
- Math Journal B: The Largest Known Prime Number (so far)
- Math Journal C: Fibonacci Connections

Journal A: Divisibility Trick for Seven

You may have noticed that the table of divisibility rules in your textbook skipped the number 7. Does that mean there is no rule for divisibility by 7? As a matter of fact, a rule for divisibility by 7 does exist, but it's a little too complicated to list in the table.

Watch Dr. James Tanton's video, "Divisibility Rule for 7." The first half of the video demonstrates the rule and the second half proves why it works. Learn the trick and try to follow the algebra in the explanation for why it works.

https://www.youtube.com/watch?v=LPgAK7whEuw

Briefly describe in your own words, the steps needed to test a number for divisibility by 7. Write down any six-digit number and, using the rule, test it for divisibility by 7. Be sure to show your steps.

Journal B: The Largest Known Prime Number (so far)

Many mathematicians throughout history, including Marin Mersenne and Sophie Germain, have devoted their lives to searching for prime numbers through painstaking calculations done by hand. Mathematicians of today continue the search for new prime numbers, but with modern computer technology, the playing field has greatly changed. Since the beginning of the 21st century, prime numbers have made news headlines again and again as the title of "largest prime number" has passed from one unimaginably large number to the next.

While the study of prime numbers may seem frivolous, the fact is that they are essential to our digital world. Prime numbers form the basis for encryption of information passed through the Internet, including credit card and banking information.

For this assignment, research a news story announcing the largest prime number to date and write a brief summary of your findings as if you were explaining it to a friend. Some questions you might consider: What is the largest prime number? When was this number discovered? What tool was used in its discovery? How many digits does the number have? Is there anything else special about this number? Can humans even comprehend a number of that size? Be sure to cite your reference article.

Journal C: Fibonacci Connections

The Fibonacci sequence is a famous sequence of numbers with many applications. Watch Scishow's video, "The Fibonacci Sequence: Nature's Code," at http://youtu.be/wTlw7fNcO-0. Write a 2–3 paragraph journal entry summarizing what you learned about Leonardo de Pisa, the Fibonacci sequence, the Golden Ratio, and how all of this relates to nature.

Activities

Complete both activities below.

- Activity A: The Sieve of Eratosthenes
- Activity B: Fibonacci Inspiration

Activity A: The Sieve of Eratosthenes

You have already learned that a prime number is a natural number greater than 1 that has exactly two natural number factors, namely 1 and itself. You also learned that a composite number is a natural number greater than 1 that has more than two natural number factors.

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"The moving power of mathematical invention is not reasoning but imagination."

Augustus de Morgan

Suppose we want to find all of the prime numbers between 1 and 100. We could examine each number between 1 and 100, trying to find factors as we did in Section 3.1, but that would be very tedious and time-consuming. Good news: there is a better way!

The Greek mathematician Eratosthenes came up with a method of finding prime numbers, which we call today the Sieve of Eratosthenes. A sieve is a strainer or sifter that lets small items pass through while catching big items, just like a kitchen colander. In this case, the sieve is filtering out (crossing out) composite numbers and keeping (circling) prime numbers. Eratosthenes' method uses a chart of numbers and the instructions below. Follow the instructions, crossing out and circling numbers on the chart as indicated, and then answer the questions.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Step 1: Cross out the number 1. (Remember, 1 is neither prime nor composite!)

Step 2: Circle 2 because it is a prime number. Then cross out every multiple of 2 on the chart. (Why? Because any number that is a multiple of 2 has 2 as a factor and is, therefore, composite.) This eliminates all of the even numbers except for 2.

Step 3: Circle 3 because it is the next prime number. Then cross out every multiple of 3 on the chart, as these numbers will also be composite.

Step 4: Circle 5 because it is the next prime number. Then cross out every multiple of 5 on the chart.

Step 5: Continue in this way (circle 7 and cross out its multiples, and so on) until every number from 1 to 100 has either been circled or crossed out. The circled numbers are all the prime numbers from 1 to 100.

After you have done that, answer the following questions.

1. Why did the instructions have you skip checking the number 4 and its multiples?
2. List all of the prime numbers between 1 and 100. How many primes are there between 1 and 100?
3. When exploring mathematics, we always look for patterns. One pattern you may have noticed is that for each number whose multiples you crossed out, you did not have to cross out any multiples until you reached the square of that number. For example, when you crossed out multiples of 3, you circled 3 because it was prime, but 6 was already crossed out. $3^2 = 9$ was the first multiple that needed to be crossed out. Likewise, when you crossed out multiples of 5, the multiples 10, 15, and 20 were already crossed out, leaving $5^2 = 25$ as the first multiple to be crossed out. Why is this the case? (Hint: think about what numbers are divisors of the numbers already crossed out.)
4. What was the largest prime number from 1 to 100 that had a multiple that still needed to be crossed out? In other words, after which prime number could you stop checking for multiples? How did you know for certain that there would be no more primes between 1 and 100? Try to base your explanation on the pattern mentioned in question 3.
5. Suppose you were asked to use the Sieve of Eratosthenes to find all prime numbers between 1 and 400. What would be the largest prime number you would need to check before you could stop crossing out composite numbers and declare all prime numbers found?
6. Apply what you have learned through this activity to find the next five prime numbers greater than 100.

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7. An *emirp* (prime written backward) is a prime number whose digits can be reversed and form a different prime number. The number 13 is the smallest emirp. The reverse of its digits is 31, which is also a prime number. What other emirps do you see on your chart of prime numbers?

8. If a prime number is doubled and increased by one, and the result is a prime number, then the result 2p+1 is called a Sophie Germain prime. For example, 11 is a Sophie Germain prime because it is the result when the prime number 5 is doubled and increased by 1, but the prime number 19 is not a Sophie Germain prime because 19=2(9)+1 and 9 is not prime. Select any three prime numbers under 100 and test to see if they are Sophie Germain primes. Show your steps.

Activity B: Fibonacci Inspiration

Now that you have learned about the Fibonacci sequence, watch Cristóbal Vila's short video, "Nature by Numbers," which visually explores some of the connections of the Fibonacci sequence to nature.

https://www.youtube.com/watch?v=kkGeOWYOFoA

Afterward, reflect on which aspect of the video you found most interesting and, using that as inspiration, create a poem, short story, drawing, painting, choreographed dance, original musical composition, or any other creative work. Include a brief written explanation of how your creative piece features the connection between nature and the Fibonacci sequence. For additional inspiration, check out the links on the Math Connections resources page on the Oak Meadow website.

Extra credit opportunity for enrolled students: For extra credit, participate in a discussion with other students in this course. You can access the discussion thread by going to MyMathLab and navigating to the chapter 3 page. Share your creative piece, if you'd like to, or simply offer your thoughts on this video or one of the additional resource videos on the Fibonacci sequence. Be sure to also respond to other students to get a conversation going. If you know of other Fibonacci or Golden Ratio resources your classmates might enjoy, feel free to suggest them! If you enjoy interacting with your fellow students, you are encouraged to post in the group on other math topics throughout this course.

Going Further: Additional Resources

For additional resources on the Fibonacci sequence, the Sieve of Eratosthenes, scientific notation, exponents, and more, visit the resource page on the Oak Meadow website.

For Enrolled Students

Once you have finished lesson 3, please submit the following items to your teacher:

- Journal entries A, B, and C
- Activities A and B (let your teacher know if you completed the optional part of Activity B for extra credit)
- Lesson 3 Test (from the test packet)

Make sure your submission is complete before sending it to your teacher.

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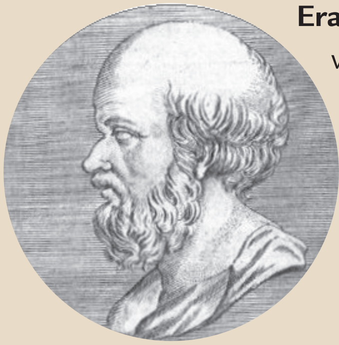
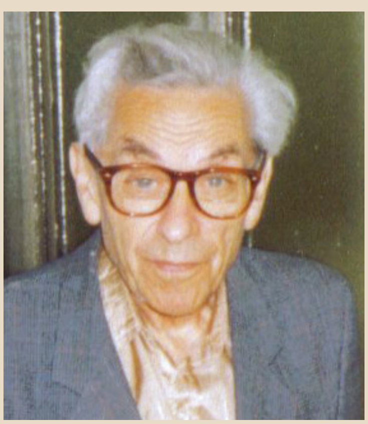
People in Mathematics



Eratosthenes (276–194 BCE), who was born in Northern Africa, was a scholar and librarian at the Library of Alexandria in Egypt. Among his many mathematical and geographical contributions, he calculated with impressive accuracy the circumference of the Earth by comparing the lengths of shadows in two different cities. He also worked extensively with prime numbers and his Sieve of Eratosthenes is still used today in number theory.

Sophie Germain (1776–1831) was so determined to become a mathematician despite opposition from her parents that as a teenager, she taught herself to read Latin and Greek and endured cold nights huddled under blankets with her math books. Her family of wealthy French merchants, who did not approve of women studying, had confiscated her light, fire, and clothing in an effort to make her stop learning. Young Sophie persevered and submitted a paper under a pseudonym to the well-respected mathematician Lagrange. He was so impressed that he searched for the paper's author, and when he discovered Sophie had written it, he decided to sponsor her. She spent much of her career working on prime numbers and Fermat's Last Theorem.



Paul Erdös (1913–1996) was a child prodigy born to a Jewish Hungarian family. He grew up in the midst of World War I and fled to the United States in World War II. He loved numbers and mathematics to the point that he could not be bothered with anything else, even taking care of himself. He spent most of his life traveling around, living as a guest of other mathematicians and collaborating on many problems in number theory and combinatorics. In fact, he wrote or contributed to an incredible 1,475 math papers, and his fans developed a system of "Erdös numbers" to describe a person's academic relationship to him for bragging purposes. A person with Erdös number 1 published a paper with Erdös, while a person with Erdös number 2 published a paper with a person who published a paper with Erdös, and so on.