Calculus

Calculus treats all the topics normally covered in an Advanced Placement AB-level calculus program, as well as many of the topics required for a BC-level program. The text begins with a thorough review of those mathematical concepts and skills required for calculus. In the early problem sets, students practice setting up word problems they will later encounter as calculus problems. The problem sets contain multiple-choice and conceptually-oriented problems similar to those found on the AP Calculus examination. Whenever possible, students are provided an intuitive introduction to concepts prior to a rigorous examination of them. Proofs are provided for all important theorems.

Prerequisite: Advanced Math.

The following books are needed for this course:

Calculus (Saxon)

Contents of Saxon *Calculus with Trigonometry and Analytic Geometry* (textbook and Home Study Packet):

15. Evaluating polynomials
16. Continuity; Left-hand and right-hand limits
17. Sum and difference identities for trigonometric functions; Double-angle identities for sine and cosine
18. Graphs of logarithmic functions; Period of a function
19. Limit of a function
20. The parabola as a locus; Translated parabolas
21. Inverse trigonometric functions; Trigonometric equations
22. Interval notation; Products of linear factors; Tangents; Increasing and decreasing functions
23. Logarithms of products and quotients; Logarithms of powers; Exponential equations
24. Infinity as a limit; Undefined limits
25. Sums, products, and quotients of functions; Composition of functions
26. Locus development; Equation of the ellipse; Foci
27. The derivative
28. Change of base; Logarithmic inequalities
29. Translation of functions; Rational functions I
30. The hyperbola
31. Binomial expansion; Recognizing the equations of conic sections
32. Roots of complex numbers; Trigonometric functions of $n\theta$
33. The derivative of x^n; Notations for the derivative
34. Identities for the tangent function; Area and volume
35. The constant-multiple rule; The derivatives of sums and differences
36. Exponential growth and decay x
37. Derivative of e^x and $\ln |x|$; Derivative of $\sin x$ and $\cos x$
38. Equation of the tangent line; Higher order derivatives
39. Graphs of rational functions II; A special limit
40. Newton and Leibnitz; The differential
41. Graph of $\tan \theta$; Graphs of reciprocal functions
42. Product rule for derivatives and differentials; Proof of the product rule
43. An antiderivative; Integration
44. Factors of polynomial functions; Graphs of polynomial functions
45. Implicit differentiation
46. The integral of a constant; Integral of $Cf(x)$; Integral of x^n
47. Critical numbers
48. Differentiation by u substitution
49. Integral of a sum; Integral of $1/x$
50. Units for the derivative; Normal lines
51. Graphs of rational functions III; Repeated factors
52. The derivative of a quotient; Proof of the quotient rule
53. Area under a curve
54. The chain rule; Equivalent forms for the derivative
55. Using f' to characterize f; Using f' to define max and min
56. Related rate problems
57. Fundamental theorem of integral calculus
58. Derivatives of trigonometric functions; Summary of rules for derivatives and differentials

59. Concavity and inflection points; Applications of the second derivative
60. Derivatives of composite functions; Derivatives of products and quotients
61. Integration by guessing
62. Maximization and minimization problems
63. Riemann sum; The definite integral
64. Velocity and acceleration (motion I); Motion due to gravity
65. More integration by guessing
66. Properties of the definite integral
67. Explicit and implicit equations; Inverse functions
68. Computing areas
69. Area between two curves
70. Game playing with f, f', and f''
71. Applications of the definite integral I
72. Critical number (closed interval) theorem
73. Derivatives of inverse trigonometric functions; What to memorize
74. Falling body problems
75. *U* substitution; Change of variable; Proof of the substitution theorem
76. Functions of y
77. Even and odd functions
78. Integration by parts
79. Properties of limits; Some special limits
80. Solids of revolution
81. Derivatives and integrals of a^x and $\log_a x$; Derivative of $|x|$
82. Fluid force
83. Continuity of functions
84. Integration of odd powers of sin x and cos x
85. Applications of the definite integral (work II)
86. Particle Motion III
87. l'Hopital's rule; Proof of l'Hopital's rule
88. Asymptotes of rational functions
89. Balance points
90. Volume by washers
91. Limits and continuity; Differentiability
92. Integration of even powers of sin x and cos x
93. Centroids
94. Logarithmic differentiation
95. The mean value theorem; Application of the mean value theorem; Proof of Rolle's theorem
96. Rules for even and odd functions
97. Volume by shells
98. Separable differential equations
99. Average value of a function; Mean value theorem for integrals
100. Particle motion IV
101. Derivatives of inverse functions
102. Solids of revolution IV

103. Absolute value
104. Integral of tan”*x*; Integral of cot”*x*
105. Second fundamental theorem of integral calculus; The natural logarithm function
106. Approximation with differentials
107. Limit of sin *x*/*x*; A note (optional)
108. Integrals of sec *u* and csc *u*; Trig substitution
109. Polar equations; Polar graphing
110. Partial fractions I
111. Polar graphing II
112. Partial fractions II
113. Integration by parts II
114. Implicit differentiation II
115. Partial fractions II
116. Derivative of e^x and of ln *x*; Derivative of sin *x*
117. Proofs of the fundamental theorem; Epsilon delta proofs

Lesson 15

Evaluating Polynomials

15A. The Remainder Theorem.

To evaluate polynomials, we can make use of the remainder theorem.

> THE REMAINDER THEOREM
>
> If a constant c is a zero of a polynomial and if we divide the polynomial by $x - c$, the remainder will be zero. If the remainder is not zero, the remainder equals the value of the polynomial when x equals c.

To look at a specific example, we will manufacture a cubic polynomial by multiplying the factors $x + 1$, $x - 2$, and $x + 3$, as we show on the left below. Since one of the factors of the polynomial is $x - 2$, then $x - 2$ will divide the polynomial evenly and will have no remainder, as we show on the right.

$$f(x) = (x + 1)(x - 2)(x + 3)$$
$$f(x) = x^3 + 2x^2 - 5x - 6$$

$$\begin{array}{r|rrrr}
 & x^2 + & 4x & + & 3 \\ \hline
x - 2 & x^3 + & 2x^2 & - 5x & - 6 \\
 & x^3 - & 2x^2 & & \\ \hline
 & & 4x^2 & - 5x & \\
 & & 4x^2 & - 8x & \\ \hline
 & & & 3x & - 6 \\
 & & & 3x & - 6 \\ \hline
 & & & & 0
\end{array}$$

The value of a polynomial is zero if x is replaced by a number that is a zero of the polynomial. The factored form and the unfactored form of the f polynomial of this example are equivalent expressions, and both will equal zero if x is replaced by 2.

$$\begin{aligned} f(2) &= (2 + 1)(2 - 2)(2 + 3) \\ &= (3)(0)(5) \\ &= 0 \end{aligned} \qquad \begin{aligned} f(2) &= (2)^3 + 2(2)^2 - 5(2) - 6 \\ &= 8 + 8 - 10 - 6 \\ &= 0 \end{aligned}$$

Now, if we add 10 to the f polynomial, we get a new polynomial that we call the g polynomial. The g polynomial does not have $x - 2$ as a factor. When we divide the g polynomial by $x - 2$, we get a remainder of 10.

$g(x) = (x + 1)(x – 2)(x + 3) + 10$
$g(x) = x^3 + 2x^2 – 5x + 4$

$$
\begin{array}{r|rrrr}
 & x^2 + & 4x & + & 3 \\ \hline
x-2 & x^3 + & 2x^2 & - 5x & + 4 \\
 & x^3 - & 2x^2 & & \\ \hline
 & & 4x^2 & - 5x & \\
 & & 4x^2 & - 8x & \\ \hline
 & & & 3x & + 4 \\
 & & & 3x & - 6 \\ \hline
 & & & & 10
\end{array}
$$

Both the factored and unfactored forms of the *g* equation have a value of 10 when *x* is replaced with 2.

$g(2) = (2 + 1)(2 – 2)(2 + 3) + 10$
$= (3)(0)(5) + 10$
$= 10$

$g(2) = (2)^3 + 2(2)^2 – 5(2) + 4$
$= 8 + 8 –10 + 4$
$= 10$

Thus, if we have a polynomial and divide by $x – 11$, we can find out one of two things by looking at the remainder. If the remainder is zero, then 11 is a zero of the polynomial. If the remainder is not zero, the remainder equals the value of the polynomial when $x = 11$.

<u>15B. Synthetic Division.</u>

Because dividing a polynomial by $x – c$ gives us such useful information, synthetic division was invented to permit this type of division to be done quickly. Synthetic division can be used to divide a polynomial by $x – 4$ or $x + 3$, but cannot be easily used to divide by forms such as $2x + 4$ or $x^2 – 5$. Both the coefficient and the exponent of *x* in the divisor must be 1 for this version of synthetic division to be used. To evaluate a polynomial when $x = c$, the division is a process of bringing down the lead coefficient, multiplying, adding, multiplying, adding, etc.

Example 15.1 Use synthetic division to divide $2x^3 + 3x^2 – 4x – 7$ by $x + 1$.

Solution To divide by $x + 1$, we use -1 as a divisor and write down only the constants in the dividend. The first step is to bring down the first constant, which is 2.

$$
\begin{array}{r|rrrr}
-1 & 2 & 3 & -4 & -7 \\
 & \downarrow & & & \\ \hline
 & 2 & & &
\end{array}
$$

The next step is to multiply 2 by the divisor, -1, for a product of -2, which we record under the 3.

-1	2	3	-4	-7
	↓	-2		
	2			

Then we add 3 and -2 to get 1. We multiply 1 by the divisor, -1, and record the product under the -4. Then we add, multiply, record, and add again.

-1	2	3	-4	-7
	↓	-2	-1	5
	2	1	-5	-2

The first three numbers below the line are the coefficients of the reduced polynomial, and the remainder is -2. The greatest exponent of the reduced polynomial is always 1 less than the greatest exponent of the dividend. Thus we may write

$$2x^3 + 3x^2 - 4x - 7 = (x + 1)(2x^2 + x - 5) -2$$

The remainder -2 is the value of the polynomial if x = -1. We can see this because the first factor, $(x + 1)$, of the factored form equals zero if x *equals* -1.

Example 15.2 Use synthetic division to evaluate $3x^3 - 4x - 2$ when $x = 2$.

Solution We enter the coefficients and remember to write a zero coefficient for x^2.

2	3	0	-4	-2
	↓	6	12	16
	3	6	8	14

The numbers 3, 6, and 8 are the coefficients of the reduced polynomial and 14 is the remainder. Thus, the value of $3x^3 - 4x - 2$ when $x = 2$ is 14; so 2 is not a zero of the polynomial.

$$3x^3 - 4x - 2 = (3x^2 + 6x + 8)(x-2) + 14$$

15C. Rational Zero (Root) Theorem.

If a number is a zero of a polynomial, the number is also a root of the polynomial equation formed by setting the polynomial equal to zero. We know that we can find rational and irrational zeros of quadratic polynomials by completing the square or by using the quadratic formula. We remember that the rational zero theorem allows us to list the possible rational zeros of any polynomial, but does not tell which, if any, of these possible zeros really is a zero. There is no comparable theorem for irrational zeros. This is unfortunate because most of the polynomial equations encountered in real-world problems have irrational roots. Before computers were invented, much time was spent laboriously finding rational roots and estimating irrational roots. To prevent this waste of time, most of the polynomials encountered in this book will be contrived to have integral zeros of ±1, ±2, or ±3. The rational zero theorem is useful in helping us decide which of these roots to investigate.

We remember that the rational zero (root) theorem tells us that if a polynomial whose coefficients are integers has a rational zero, then this number is a fraction whose numerator is some integral factor of the constant term and whose denominator is some integral factor of the lead coefficient. Thus, the possible rational zeros of $4x^{14} + 3x^8 + 7x + 3$ can be found by forming a fraction by using one of the integer factors of 3 above the line as a numerator.

$$\frac{\text{Factors of 3}}{\text{Factors of 4}} = \frac{1, -1, 3, -3}{1, -1, 2, -2, 4, -4}$$

and one of the integer factors of 4 below the line as a denominator. If we do this, we can list the possible rational zeros as

$$1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{3}{4}, -\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{2}, -\frac{3}{2}, 3, -3$$

Example 15.3 Find the zeros of the function $f(x) = x^3 + 2x^2 - 2x - 4$.

Solution We could use a computer to estimate the zeros, but since this is a problem in a beginning calculus book, we hope the author put at least one rational zero in the polynomial. If so, the rational root can be found from the following list

$$\frac{+1, -1, +2, -2, +4, -4}{+1, -1} \rightarrow 1, -1, 2, -2, 4, -4$$

We will use synthetic division to see which, if any, of these numbers are zeros.

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -2 & -4 \\ & & 1 & 3 & 1 \\ \hline & 1 & 3 & 1 & -3 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & 2 & -2 & -4 \\ & & -1 & -1 & 3 \\ \hline & 1 & 1 & -3 & -1 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -2 & -4 \\ & & 2 & 8 & 12 \\ \hline & 1 & 4 & 6 & 8 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 2 & -2 & -4 \\ & & -2 & 0 & 4 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

The last remainder is zero; so -2 is a zero of the polynomial function and $x + 2$ is a factor of the polynomial. Now we have

$$x^3 + 2x^2 - 2x - 4 = (x + 2)(x^2 - 2)$$

We will find the roots of the equation $x^2 - 2 = 0$. These numbers are the zeros of the polynomial $x^2 - 2$.

$$x^2 - 2 = 0 \rightarrow x^2 = 2 \rightarrow \boldsymbol{x = \pm\sqrt{2}}$$

Thus, the factors of the original polynomial are $(x + 2)$, $(x - \sqrt{2})$, and $(x + \sqrt{2})$, and the zeros are **-2, √2, -√2**.

Example 15.4 Find the roots of $x^3 - 7x - 6 = 0$.

Solution First we list the possible rational roots.

$$+1, -1, +2, -2, +3, -3, +6, -6$$

Now we divide. Note that we write a zero for the coefficient of the x^2 term in the dividend.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & -6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & -12 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Zero is the remainder when we divide by -1, so -1 is a zero of the polynomial. The reduced polynomial is $x^2 - x - 6$, and we can use the quadratic formula to find the zeros of this polynomial.

$$x = \frac{1 \pm \sqrt{1-(4)(1)(-6)}}{2} = \frac{1 \pm \sqrt{25}}{2} = -2, 3$$

Thus the roots of the polynomial equation are **-2, -1, and 3**.